

HOLOMORPHIC ISOMETRIC EMBEDDINGS OF THE PROJECTIVE LINE INTO QUADRICS

OSCAR MACIA, YASUYUKI NAGATOMO, MASARO TAKAHASHI

ABSTRACT. We discuss holomorphic isometric embeddings of the projective line into quadrics using the generalisation of the theorem of do Carmo–Wallach in [10] to provide a description of their moduli spaces up to image and gauge–equivalence. Moreover, we show rigidity of the real standard map from the projective line into quadrics.

1. INTRODUCTION

A harmonic map from a Riemannian manifold into a Grassmannian manifold is characterized by a triple composed by (a) a vector bundle, (b) a space of sections of this bundle and (c) a Laplace operator, as shown in [10]. This characterisation can be regarded as a generalisation of a theorem of T. Takahashi [11] which proves that an isometric immersion of a Riemannian manifold in Euclidean space is an eigenvector for the Laplacian iff it is a minimal immersion in some Euclidean sphere, the energy density being related to the corresponding eigenvalue. In its generalised form the standard sphere in Euclidean space is identified with the appropriate Grassmannian, and the isometric immersion (the position vector) is considered as a section of the universal quotient bundle over it. Then, the energy density of the mapping is related to the mean curvature operator (defined in [10], §2) of the pull–back of the universal quotient bundle. Hence, the theorem of Takahashi can be reformulated from the viewpoint of vector bundles and their spaces of sections, leading to the characterisation through the aforesaid triple.

In this viewpoint, a vector bundle and a finite dimensional space of sections induce a map into a Grassmannian. A celebrated example of such induced map is Kodaira’s embedding of an algebraic manifold into complex projective space [9], which is induced by a holomorphic line bundle and the space of holomorphic sections.

The pioneering work of Takahashi found application in do Carmo and Wallach undertaking of the classification of minimal (isometric) immersions of spheres into spheres [4]. Their staggering result reveals that starting at dimension three and constant scalar curvature four, there are continuous families of image inequivalent

minimal immersions of spheres into standard spheres of dimension high enough. Each of these families is parametrised by a moduli space depending on no less than eighteen parameters, yielding a lower bound for the moduli dimension. Alternatively, for the lower dimension and constant scalar curvature cases minimal immersions of spheres into spheres are unique whenever they exist. In this case, the moduli collapses to a point and the map is said to be rigid.

A key role in do Carmo–Wallach theory is played by a symmetric semipositive–definite linear operator interweaving minimal immersions. Finding the space of the image inequivalent operators amounts to describe the moduli space, an endeavour which is dealt successfully with representation–theoretic techniques.

The generalisation of the theorem of Takahashi in [10] allowed the second–named author to achieve altogether a generalisation of the results of do Carmo–Wallach. In its general form the induced map into a Grassmannian by the aforementioned harmonic triple is naturally equipped with a family of Hermitian semipositive–definite operators determining the moduli space.

Uniqueness of the associated Hermitian operator reduces the moduli to a single point granting rigidity of the induced map. An important illustration of such behaviour is Bando and Ohnita’s result [1] stating the rigidity of the minimal immersion of the complex projective line into complex projective spaces, originally proved employing twistor methods. Another instance is rigidity of holomorphic isometric embeddings between complex projective spaces, which is part of Calabi’s result [2]. Both conclusions can be given a unified treatment applying the generalisation of the theorem of do Carmo–Wallach.

Closer to the vector bundle viewpoint is Toth’s analysis of polynomial minimal immersions between projective spaces [12] where the spaces of harmonic polynomials in complex space are used to define polynomial maps between spheres and the Hopf fibration to get a map between complex projective spaces. Representation theory of unitary groups is then put in practice to determine a lower bound for the moduli dimension.

It is remarkable that neither [4] nor [12] do require the vector bundle viewpoint of [10] since in this later sense respond for straightforward situations: in the original do Carmo and Wallach construction the associated vector bundle would be the trivial bundle; Toth’s result follows from considering a complex line bundle with canonical connection.

The study of harmonic maps from the complex projective line into complex quadrics has previously been pursued in different ways, e.g., in [3], [6], [7] and [13]. In the present article we apply the generalisation of the theorem of do Carmo–Wallach to the study of holomorphic isometric embeddings of the projective line into quadrics to give a description of the moduli spaces up to image and

gauge-equivalence. The authors would like to emphasize that contrary to [4, 12] their approach allows to compute the exact dimension of the moduli spaces. Also notice that by considering complex quadrics as target instead of complex projective spaces as in [2], positive-dimensional moduli spaces appear.

The article is organised as follows: In §2 we introduce the required preliminaries to the theory culminating in the statement of the generalisation of the theorem of do Carmo–Wallach (theorem 2.4) as developed in [10]. The following two sections are technical in nature. First, §4 gives an account of certain relevant spectral formulae for real $SU(2)$ representations. After that, §5 deals with the study of the space of Hermitian/symmetric operators and banking on §4, concludes with a detailed description of its various subspaces (proposition 5.4) which encode the information about the moduli spaces. Applications of the theory first appear in §6: by showing that the space of symmetric operators yielding the moduli space restricts to a single point, we prove rigidity for the real standard map (theorem 6.4). Finally, in Sections 7 and 8 the moduli spaces up to image and gauge equivalence are introduced and described (theorems 7.4 and 8.1).

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2. PRELIMINARIES

In this section we give a short account of results concerning vector bundles endowed with fibre-metrics and connections needed to state a version of the generalisation of the theorem of do Carmo–Wallach (theorem 2.4), whose implications will be applied later in this article. In essence, the theorem establishes a correspondence between *geometric* gauge-theoretic information associated to a special class of harmonic mappings and *algebraic* representation-theoretic properties of some Hermitian operators.

Let us start by introducing the required geometric background.

Let W be a complex (resp. real, resp. real oriented) N -dimensional vector space and $Gr_p(W)$ the complex (resp. real, resp. real oriented) Grassmannian p -planes in W . Generically, \underline{W} will stand for the total space of a trivial vector bundle $\underline{W} \rightarrow B$ with fibre W over some specified base manifold B . Denote by $\underline{W} \rightarrow Gr_p(W)$ the trivial bundle of fibre W over $Gr_p(W)$. Then, there is a natural

bundle injection $i_S : S \rightarrow \underline{W}$ of the *tautological vector bundle* $S \rightarrow Gr_p(W)$ into the aforementioned trivial bundle. The *universal quotient bundle* $Q \rightarrow Gr_p(W)$ is defined by the exactness of the sequence $0 \rightarrow S \rightarrow \underline{W} \rightarrow Q \rightarrow 0$. Denote by π_Q the natural projection $\underline{W} \rightarrow Q$ and use it to regard W as a subspace of $\Gamma(Q)$, the space of sections of the universal quotient bundle.

By fixing a Hermitian (resp. symmetric) inner product on W the tautological and universal quotient bundles $S, Q \rightarrow Gr_p(W)$ inherit a fibre-metric, and can be given canonical connections and second fundamental forms in the sense of Kobayashi [8].

Suppose $V \rightarrow M$ is a complex (resp. real, resp. real oriented) vector bundle of rank q and consider a N -dimensional space of sections $W \subset \Gamma(V)$. By definition of $\underline{W} \rightarrow M$, there is a bundle homomorphism $ev : \underline{W} \rightarrow V$, called *evaluation*, defined by $(x, t) \mapsto t(x)$ for all $t \in W, x \in M$. The vector bundle $V \rightarrow M$ is said to be *globally generated by W* if the evaluation is surjective. Under this hypothesis, there is a map $f : M \rightarrow Gr_p(W)$, where $Gr_p(W)$ is a complex (resp. real, resp. real oriented) Grassmannian and $p = N - q$, defined by

$$f(x) := \text{Ker } ev_x = \{t \in W \mid t(x) = 0\},$$

where $ev_x \equiv ev(x, \cdot)$. The map f is said to be *induced by* $(V \rightarrow M, W)$, or simply by W if the vector bundle $V \rightarrow M$ is specified.

Notice that, by the definition of induced map, $V \rightarrow M$ can be *naturally identified* with $f^*Q \rightarrow M$. Therefore, given a smooth map $f : M \rightarrow Gr_p(W)$, it can be regarded as the induced map determined by the couple $(f^*Q \rightarrow M, W)$. If the inclusion $W \rightarrow \Gamma(f^*Q)$ is injective, we say that the map f is *full*.

Moreover, assume M to be Riemannian and $V \rightarrow M$ to be equipped with a fibre-metric and a connection. From these data a Laplace operator acting on sections can be defined.

The model special case is that in which M is a compact reductive homogeneous space G/K (where G is a compact Lie group and K is a closed subgroup of G), and $V \rightarrow M$ is a homogeneous complex (resp. real) vector bundle of rank q , i.e., $V \cong G \times_K V_0$ where V_0 is a q -dimensional complex (resp. real) K -module. If additionally V_0 admits a K -invariant Hermitian (resp. symmetric) inner product, $V \rightarrow M$ inherits a G -invariant Hermitian (resp. symmetric) fibre-metric.

By reductivity, $V \rightarrow M$ is equipped with a canonical connection too, the one for which the horizontal subspace on the principal K -bundle $G \rightarrow M$ is given by the complement \mathfrak{m} to $\mathfrak{k} = L(K)$ in $\mathfrak{g} = L(G)$.

Using the Levi-Civita connection and the canonical connection, $\Gamma(V)$ can be decomposed into eigenspaces of the Laplacian each being a finite-dimensional not necessarily irreducible G -module and equipped with a G -invariant L^2 -inner product. Then, we say that the induced map by $(V \rightarrow M, W)$ is *standard* if a G -submodule $W \subseteq W_\mu$ globally generates the bundle, where W_μ is the eigenspace of the Laplacian with eigenvalue μ .

Evidently, the definition of standard map reaches beyond the special homogeneous case. The spaces of sections inducing standard maps have the following interesting property which will be useful later:

Lemma 2.1. [10] *Let W be a G -subspace of W_μ . If W globally generates $V \rightarrow G/K$, then V_0 can be regarded as a subspace of W .*

Denote by U_0 the orthogonal complement of V_0 in W . Then, the induced standard map $f_0 : M \rightarrow Gr_p(W)$ is expressed as

$$f_0([g]) = gU_0 \subset W,$$

for all $[g] \in G/K$, and is G -equivariant.

Notice that, besides its assumed fibre-metric and connection, $V \rightarrow M$ is endowed with a secondary couple of fibre-metric and connection inherited from the natural identification $V \cong f^*Q$, i.e., the fibre-metric and canonical connection on $Q \rightarrow Gr_p(W)$ pulled-back to $f^*Q \rightarrow M$.

In general, these structures do not need to be gauge equivalent unless the splitting $W = U_0 \oplus^\perp V_0$ satisfies extra conditions:

Lemma 2.2. [10] *The pull-back connection is gauge equivalent to the canonical connection if and only if*

$$\mathfrak{m}V_0 \subset U_0.$$

As we shall have opportunity to see, this condition will turn out to be very relevant for the development of the remaining theory.

For a standard map f , the second fundamental forms H and K of the tautological and universal quotient bundles respectively can be assembled together in the *mean curvature operator of f* , a section of $\text{End}(f^*Q)$ defined in [10], §2, as

$$A = \sum_{i=1}^n H_{df(e_i)} K_{df(e_i)}$$

where $\{e_i\}_{1 \leq i \leq n}$ is an orthonormal basis of the tangent space of M .

Lemma 2.3. [10] *If a G -module $W \subseteq W_\mu$ globally generates $V \rightarrow M$ and satisfies the condition $\mathfrak{m}V_0 \subset U_0$, then the standard map $f_0 : M \rightarrow Gr_p(W)$ is harmonic with constant energy density $e(f_0) = q\mu$ and the mean curvature operator proportional to the identity $A = -\mu Id_V$.*

Let us introduce the two increasingly stronger equivalence relations up to which we shall later define moduli spaces of maps: Let f_1 and $f_2 : M \rightarrow Gr_p(W)$. Then f_1 is called *image equivalent* to f_2 if there exists an isometry ϕ of $Gr_p(W)$ such that $f_2 = \phi \circ f_1$. Furthermore, denote by $\tilde{\phi}$ the bundle isomorphism of $Q \rightarrow Gr_p(W)$ which covers the isometry ϕ of $Gr_p(W)$. Then, the pair (f_1, ϕ_1) is said to be *gauge equivalent* to (f_2, ϕ_2) , where $\phi_i : V \rightarrow f_i^*Q$ ($i = 1, 2$) are bundle

isomorphisms, if there exists an isometry ϕ of $Gr_p(W)$ such that $f_2 = \phi \circ f_1$ and $\phi_2 = \tilde{\phi} \circ \phi_1$.

Aside from the geometric background, some algebraic preliminaries regarding Hermitian operators are needed.

Let G be a compact Lie group, W a complex (resp. real) representation of G together with an invariant Hermitian (resp. inner) product $(\cdot, \cdot)_W$ and denote by $H(W)$ (resp. $S(W)$) the set of Hermitian (resp. symmetric) endomorphisms of W . We equip $H(W)$ (resp. $S(W)$) with a G -invariant inner product $(A, B)_H = \text{trace } AB$, for $A, B \in H(W)$. Define a Hermitian (resp. symmetric) operator $H(u, v)$ (resp. $S(u, v)$) for $u, v \in W$ as

$$H(u, v) := \frac{1}{2} \{u \otimes (\cdot, v)_W + v \otimes (\cdot, u)_W\} \quad (\text{resp. } S(u, v))$$

If U and V are subspaces of W , we define a real subspace $H(U, V) \subset H(W)$ (resp. $S(U, V) \subset S(W)$) spanned by $H(u, v)$ (resp. $S(u, v)$) where $u \in U$ and $v \in V$. In a similar fashion, $GH(U, V)$ (resp. $GS(U, V)$) denotes the subspace of $H(W)$ (resp. $S(W)$) spanned by $gH(u, v)$ (resp. $gS(u, v)$).

Now we have all the needed ingredients to introduce a version of the generalisation [10] of the theorem of do Carmo–Wallach [4] for holomorphic maps which we include for convenience of the reader:

Theorem 2.4. *Let $M = G/K$ be a compact irreducible Hermitian symmetric space with decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. We fix a complex homogeneous line bundle $L = G \times_K V_0 \rightarrow G/K$ with an invariant metric h_L and the canonical connection ∇ . We regard $L \rightarrow G/K$ as a real vector bundle with complex structure J_L .*

Let $f : M \rightarrow Gr_n(\mathbf{R}^{n+2})$ be a full holomorphic map satisfying the following two conditions:

- (i) *The pull-back bundle $f^*Q \rightarrow M$ with the pull-back metric, connection and complex structure is gauge equivalent to $L \rightarrow M$ with h_L , ∇ and J_L .*
- (ii) *The mean curvature operator $A \in \Gamma(\text{End } L)$ of f is expressed as $-\mu \text{Id}_L$ with some real positive number μ , and so $e(f) = 2\mu$.*

Then we have the space of holomorphic sections W of $L \rightarrow M$ which is also an eigenspace of the Laplacian with eigenvalue μ equipped with L^2 -inner product $(\cdot, \cdot)_W$ induced from L^2 -Hermitian inner product. Regard W as a real vector space with $(\cdot, \cdot)_W$. Then, there exists a semipositive symmetric endomorphism $T \in \text{End}(W)$ such that the pair (W, T) satisfies the following four conditions:

- (I) *The vector space \mathbf{R}^{n+2} is a subspace of W with the inclusion $\iota : \mathbf{R}^{n+2} \rightarrow W$ preserving the orientation and $L \rightarrow M$ is globally generated by \mathbf{R}^{n+2} .*
- (II) *As a subspace, $\mathbf{R}^{n+2} = \text{Ker } T^\perp$ and the restriction of T is a positive symmetric transformation of \mathbf{R}^{n+2} .*

(III) The endomorphism T satisfies

$$(2.1) \quad (T^2 - Id_W, \text{GH}(V_0, V_0))_H = 0, \quad (T^2, \text{GH}(\varrho(\mathfrak{m})V_0, V_0))_H = 0.$$

(IV) The endomorphism T provides a holomorphic embedding of $Gr_n(\mathbf{R}^{n+2})$ into $Gr_{n'}(W)$, where $n' = n + \dim \text{Ker } T$ and also provides a bundle isomorphism $\phi : L \rightarrow f^*Q$.

Then, $f : M \rightarrow Gr_p(\mathbf{R}^{n+2})$ can be expressed as

$$(2.2) \quad f([g]) = (\iota^* T \iota)^{-1} \left(f_0([g]) \cap \text{Ker } T^\perp \right),$$

where ι^* denotes the adjoint operator of ι under the induced inner product on \mathbf{R}^{n+2} from $(\cdot, \cdot)_W$ on W and f_0 is the standard map by W . Such two pairs (f_i, ϕ_i) , $(i = 1, 2)$ are gauge equivalent if and only if $\iota_1^* T_1 \iota_1 = \iota_2^* T_2 \iota_2$, where (T_i, ι_i) correspond to f_i ($i = 1, 2$) under the expression in (2.2), respectively.

Conversely, suppose that a vector space \mathbf{R}^{n+2} , the space of holomorphic sections $W \subset \Gamma(V)$ regarded as real vector space and a semipositive symmetric endomorphism $T \in \text{End}(W)$ satisfying conditions (I), (II) and (III) are given. Then we have a unique holomorphic embedding of $Gr_n(\mathbf{R}^{n+2})$ into $Gr_{n'}(W)$ and the map $f : M \rightarrow Gr_n(\mathbf{R}^{n+2})$ defined by (2.2) is a full holomorphic map into $Gr_n(\mathbf{R}^{n+2})$ satisfying conditions (i) and (ii) with bundle isomorphism $L \cong f^*Q$.

Proof. This is obtained by a combination of theorems 5.16 and 5.20 in [10]. \square

Remark 1. Conditions (i) and (ii) in the theorem are named respectively gauge and Einstein–Hermitian conditions; the later is often denoted simply as EH condition for short.

3. GENERALIZED DO CARMO–WALLACH THEORY FOR HOLOMORPHIC ISOMETRIC EMBEDDINGS

The aim of this section is to introduce holomorphic isometric embeddings from \mathbf{CP}^1 into $Gr_n(\mathbf{R}^{n+2})$ and to show that they satisfy the hypothesis of the generalized version of do Carmo–Wallach theory developed in [10] and given in theorem 2.4.

We realize a complex quadric of \mathbf{CP}^{n+1} as a real oriented Grassmannian $Gr_n(\mathbf{R}^{n+2})$. Then the universal quotient bundle has a holomorphic bundle structure. Notice that the curvature two-form R of the canonical connection on the quotient bundle is the fundamental two-form ω_Q on $Gr_n(\mathbf{R}^{n+2})$ up to a constant multiple

$$R = -2\pi\sqrt{-1}\omega_Q.$$

Denote by ω_0 the fundamental two-form on \mathbf{CP}^1 . When R_1 denotes the curvature two-form of the canonical connection on $\mathcal{O}(1) \rightarrow \mathbf{CP}^1$, we also have $R_1 = -2\pi\sqrt{-1}\omega_0$

Definition 1. Let $f : \mathbf{CP}^1 \rightarrow Gr_n(\mathbf{R}^{n+2})$ be a holomorphic embedding. Then f is called an isometric embedding of degree k if $f^*\omega_Q = k\omega_0$ (and so, k must be a positive integer).

Lemma 3.1. *Let $f : \mathbf{CP}^1 \rightarrow Gr_n(\mathbf{R}^{n+2})$ be a holomorphic embedding. Then f is an isometric embedding of degree k if and only if the pull-back bundle $f^*Q \rightarrow \mathbf{CP}^1$ with the pull-back connection is gauge equivalent to $\mathcal{O}(k) \rightarrow \mathbf{CP}^1$ with the canonical connection.*

Proof. If the degree of the isometric embedding f equals k , the pull-back of the universal quotient bundle is holomorphically isomorphic to the holomorphic line bundle of degree k on \mathbf{CP}^1 (by uniqueness of the holomorphic bundle structure), which by homogeneity admits a unique Einstein–Hermitian structure up to homotheties of the fibre–metric. Uniqueness of the Einstein–Hermitian connection yields the result.

Conversely, if the pull-back of the universal quotient bundle is holomorphically isomorphic as Einstein–Hermitian bundle to the holomorphic line bundle, the pull-back fibre–metric and the Einstein–Hermitian connection coincide up to homothety, and the statement in the lemma follows. \square

Lemma 3.2. *Let $f : \mathbf{CP}^1 \rightarrow Gr_n(\mathbf{R}^{n+2})$ be a holomorphic isometric embedding of degree k . Then, the mean curvature operator $A \in \Gamma(V)$ of f is the identity on V up to a negative real constant.*

Proof. It is well-known that every holomorphic section t of $\mathcal{O}(k) \rightarrow \mathbf{CP}^1$ satisfies $\Delta t - K_{EH}t = 0$, where the Laplacian is defined through a compatible connection; K_{EH} is the mean curvature operator arising from the Hermitian structure in the sense of Kobayashi [8]. Since the canonical connection is the Einstein–Hermitian connection, $K_{EH} = \mu Id$.

On the other and, the generalisation of theorem of Takahashi [10] yields that $\Delta t + At = 0$ for $t \in \mathbf{R}^{n+2}$. Regard \mathbf{R}^{n+2} as a subspace of $H^0(\mathbf{CP}^1, \mathcal{O}(k))$; then it globally generates $\mathcal{O}(k) \rightarrow \mathbf{CP}^1$, therefore $K_{EH} = -A$, and the lemma follows. \square

These two lemmas amount to saying that the holomorphic embedding f is isometric iff it satisfies the gauge condition, and then the EH condition is automatically satisfied. Hence we can apply theorem 2.4 to obtain the moduli space \mathcal{M}_k of holomorphic isometric embeddings of degree k by the gauge equivalence of maps.

Remark 2. Unlike the case of holomorphic isometric embeddings, for general harmonic maps and minimal immersions the EH condition is independent of the gauge condition. We shall discuss harmonic maps and minimal immersions satisfying gauge and EH conditions in a forthcoming paper.

4. REAL REPRESENTATIONS OF $SU(2)$

Let $S^k \mathbf{C}^2$ be the k -th symmetric power of the standard, complex representation of $SU(2)$. Since \mathbf{C}^2 has an invariant quaternionic structure j , $S^{2k} \mathbf{C}^2$ inherits an invariant real structure $\sigma = j^{2k}$, while $S^{2k+1} \mathbf{C}^2$ is equipped with an induced invariant quaternionic structure j^{2k+1} . We shall denote the standard, real representation of $SO(3)$ by \mathbf{R}^3 and its l -th symmetric power by $S^l \mathbf{R}^3$.

We start by pointing out a fundamental relation between real irreducible representations of $SU(2)$ and $SO(3)$:

Lemma 4.1. *For $k \geq 2$, $S^k \mathbf{R}^3$ admits the following decomposition:*

$$S^k \mathbf{R}^3 = S_0^k \mathbf{R}^3 \oplus S^{k-2} \mathbf{R}^3$$

where

$$S_0^k \mathbf{R}^3 = (S^{2k} \mathbf{C}^2)^{\mathbf{R}}$$

is the real irreducible $SU(2)$ -representation defined as the σ -invariant real subspace of $S^{2k} \mathbf{C}^2$.

Proof. Consider the k -th symmetric product of \mathbf{R}^3 . Since $SO(3)$ admits an invariant symmetric two-tensor, $S^k \mathbf{R}^3$ is not irreducible but splits into $S_0^k \mathbf{R}^3 \oplus S^{k-2} \mathbf{R}^3$, for the canonical decomposition of the trace homomorphism. Counting dimensions leads to $\dim S_0^k \mathbf{R}^3 = 2k + 1$.

Recall the following elementary identity between real $SU(2)$ and $SO(3)$ representations: \mathbf{R}^3 is the real form of $S^2 \mathbf{C}^2$ induced by the involution σ , denoted by $(S^2 \mathbf{C}^2)^{\mathbf{R}}$; conversely, $S^2 \mathbf{C}^2$ is the complexification of \mathbf{R}^3 . Complexification of $S^k \mathbf{R}^3$ is given by

$$\mathbf{C} \otimes_{\mathbf{R}} S^k \mathbf{R}^3 = S^k(\mathbf{C} \otimes_{\mathbf{R}} \mathbf{R}^3) \cong S^k(S^2 \mathbf{C}^2)$$

which is a component in $\otimes^k(S^2 \mathbf{C}^2)$. Applying Clebsch–Gordan spectral formula, the top term $S^{2k} \mathbf{C}^2 \subset S^k(S^2 \mathbf{C}^2) \subset \otimes^k S^2 \mathbf{C}^2$ is a complex irreducible $SU(2)$ -representation of complex dimension $2k + 1$, equipped with an invariant real structure. Therefore, $(S^{2k} \mathbf{C}^2)^{\mathbf{R}} \subset S^k \mathbf{R}^3$ is a real irreducible $SU(2)$ -representation.

However by Schur's lemma $S^{2k} \mathbf{C}^2 \cap S^{k-2}(S^2 \mathbf{C}^2) = \{0\}$, the second factor being the complexification of $S^{k-2} \mathbf{R}^3 \subset S^k \mathbf{R}^3$, and the result follows. \square

Once we have identified the real irreducible representations of $SU(2)$ we would like to have a spectral formula for the tensor product. To that end, it is enough to restrict to the real stable subspace of the real structure

Lemma 4.2. *When $k \geq l$, then*

$$(4.1) \quad S_0^k \mathbf{R}^3 \otimes S_0^l \mathbf{R}^3 = \bigoplus_{r=0}^{2l} S_0^{k+l-r} \mathbf{R}^3.$$

Proof. Complexification of $S_0^k \mathbf{R}^3$ is given by $S^{2k} \mathbf{C}^2$. The spectral formula for the tensor product of complex irreducible representations of $SU(2)$ is well known,

$$(4.2) \quad S^{2k} \mathbf{C}^2 \otimes_{\mathbf{C}} S^{2l} \mathbf{C}^2 = \bigoplus_{r=0}^{\min(k,l)} S^{2k+2l-2r} \mathbf{C}^2$$

restricting to the real stable subspaces proves the lemma. \square

Regardless of the existence of a real structure any complex irreducible $SU(2)$ -representation can be interpreted as a real not necessarily irreducible representation by considering its underlying \mathbf{R} -vector space such that $(S^n \mathbf{C}^2)|_{\mathbf{R}} \cong (\mathbf{R}^{2n+2}, J)$ where J is a complex structure.

In the absence of a real structure, i.e., for odd n , this is a real irreducible $SU(2)$ -representation; when n is even, this is a real reducible $SU(2)$ -representation further splitting into the stable subspaces for the action of the induced real structure $(S^n \mathbf{C}^2)^{\mathbf{R}}$, which in this case are the truly real irreducible representations of $SU(2)$.

For later use in the study of real standard maps, it will be useful to have a spectral formula for the decomposition of tensor products of the underlying \mathbf{R} -vector spaces of a given complex $SU(2)$ -representations into real irreducible ones. This is how such result can be achieved: Given \mathbf{R} -vector spaces U, V underlying some complex irreducible $SU(2)$ -representations (therefore not necessarily irreducible as real representations) consider their complexification $U^{\mathbf{C}} := \mathbf{C} \otimes_{\mathbf{R}} U$, etc. These are complex not necessarily irreducible $SU(2)$ -representations, which easily abscond into irreducible ones. Taking the tensor product $U^{\mathbf{C}} \otimes_{\mathbf{C}} V^{\mathbf{C}}$ we might apply the known spectral formulae for decomposing products of complex $SU(2)$ -representations. The tensor product inherits a real structure, such that real irreducible representations can be constructed by the general formula

$$(4.3) \quad U \otimes_{\mathbf{R}} V = (U^{\mathbf{C}} \otimes_{\mathbf{C}} V^{\mathbf{C}})^{\mathbf{R}}.$$

Applying the preceding general argument we have the following

Lemma 4.3. *When we regard $S^{2k} \mathbf{C}^2$ as a real representation space \mathbf{R}^{4k+2} of $SU(2)$, the second symmetric power $S^2 \mathbf{R}^{4k+2}$ has the following irreducible decomposition:*

$$(4.4) \quad S^2 \mathbf{R}^{4k+2} = 3 \left(\bigoplus_{r=0}^k S_0^{2k-2r} \mathbf{R}^3 \right) \oplus \left(\bigoplus_{r=0}^{k-1} S_0^{(2k-1)-2r} \mathbf{R}^3 \right).$$

When we regard $S^{2k+1} \mathbf{C}^2$ as a real representation space \mathbf{R}^{4k+4} of $SU(2)$, the second symmetric power $S^2 \mathbf{R}^{4k+4}$ has the following irreducible decomposition:

$$(4.5) \quad S^2 \mathbf{R}^{4k+4} = 3 \left(\bigoplus_{r=0}^k S_0^{(2k+1)-2r} \mathbf{R}^3 \right) \oplus \left(\bigoplus_{r=0}^{k-1} S_0^{2k-2r} \mathbf{R}^3 \right).$$

Proof. Considered as its underlying \mathbf{R} -vector space, $S^{2k}\mathbf{C}^2$ is simply \mathbf{R}^{4k+2} . This is not an irreducible since $S^{2k}\mathbf{C}^2$ is equipped with an invariant real structure, but decomposes as two copies of $S_0^k\mathbf{R}^3$ by lemma 4.1. Therefore, using the spectral formula in lemma 4.2,

$$(4.6) \quad \mathbf{R}^{4k+2} \otimes_{\mathbf{R}} \mathbf{R}^{4k+2} = 2S_0^k\mathbf{R}^3 \otimes_{\mathbf{R}} 2S_0^k\mathbf{R}^3 = 4 \bigoplus_{r=0}^{2k} S_0^{2k-r}\mathbf{R}^3.$$

In order to obtain the symmetrised tensor product we need to subtract the alternating terms $\Lambda^2\mathbf{R}^{4k+2}$.

Complexifying, $\Lambda^2 = \mathbf{C} \otimes_{\mathbf{R}} \Lambda^2\mathbf{R}^{4k+2} = \Lambda^2(S^{2k}\mathbf{C}^2 \oplus S^{2k}\overline{\mathbf{C}}^2)$. Write $\Lambda^{2,0}$ for $\Lambda^2(S^{2k}\mathbf{C}^2)$ etc. Then, $\Lambda^2 = \Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2}$ where, at the real level, $\Lambda^{2,0} \cong \Lambda^{0,2}$. Using (4.2) we easily identify $(\Lambda^{2,0} \oplus \Lambda^{0,2})^{\mathbf{R}} = 2(\bigoplus_{r=0}^{2k} S^{4k-2(2r+1)}\mathbf{C}^2)^{\mathbf{R}}$, hence from (4.1)

$$(4.7) \quad (\Lambda^{2,0} \oplus \Lambda^{0,2})^{\mathbf{R}} \cong 2 \bigoplus_{r=0}^{k-1} S_0^{2k-(2r+1)}\mathbf{R}^3$$

For $(\Lambda^{1,1})^{\mathbf{R}}$ recall that $(S^{2k}\mathbf{C}^2) \cong \mathbf{C} \otimes_{\mathbf{R}} S_0^k\mathbf{R}^3 \cong S^{2k}\overline{\mathbf{C}}^2$; therefore, substitution of $U = V = S_0^k\mathbf{R}^3$ in (4.3) leads to $(\Lambda^{1,1})^{\mathbf{R}} \cong \otimes^2 S_0^k\mathbf{R}^3$; using, (4.1):

$$(4.8) \quad (\Lambda^{1,1})^{\mathbf{R}} \cong \bigoplus_{r=0}^{2k} S_0^{2k-r}\mathbf{R}^3.$$

Subtracting (4.7) and (4.8) from (4.6) yields the desired result (4.4).

When regarding $S^{2k+1}\mathbf{C}^2$ as a \mathbf{R} -vector space, we identify it with \mathbf{R}^{4k+4} which is an irreducible $\mathrm{SU}(2)$ -representation. Its complexification is reducible under the action of the inherited complex structure leading to $(\mathbf{R}^{4k+4})^{\mathbf{C}} \cong S^{2k+1}\mathbf{C}^2 \oplus S^{2k+1}\overline{\mathbf{C}}^2$. We compute the tensor product to be:

$$(4.9) \quad \mathbf{R}^{4k+4} \otimes_{\mathbf{R}} \mathbf{R}^{4k+4} \cong 4 \bigoplus_{r=0}^{2k+1} S_0^{2k+1-r}\mathbf{R}^3$$

applying Eqns. (4.1-4.3) to simplify the expression :

$$\left((S^{2k+1}\mathbf{C}^2 \oplus S^{2k+1}\overline{\mathbf{C}}^2) \otimes_{\mathbf{C}} (S^{2k+1}\mathbf{C}^2 \oplus S^{2k+1}\overline{\mathbf{C}}^2) \right)^{\mathbf{R}}.$$

Again, to get the result we need to subtract the subspace of alternating elements. Applying the same arguments as in the previous case (where now

$\Lambda^{2,0} = \Lambda^2(S^{2k+1}\mathbf{C}^2)$, etc.), we are lead to:

$$(4.10) \quad (\Lambda^{2,0} \oplus \Lambda^{0,2})^{\mathbf{R}} = 2 \bigoplus_{r=0}^{2k} S_0^{2k-r} \mathbf{R}^3,$$

$$(4.11) \quad (\Lambda^{1,1})^{\mathbf{R}} = \bigoplus_{r=0}^{2k+1} S_0^{2k+1-r} \mathbf{R}^3.$$

Equation (4.5) is achieved by subtracting (4.10) and (4.11) from (4.9). \square

5. THE SPACE OF HERMITIAN/SYMMETRIC ENDOMORPHISMS OF $SU(2)$ –REPRESENTATIONS

In the previous section we summarised some results on real $SU(2)$ –modules but in order to apply the generalised do Carmo–Wallach theory we need a deeper understanding of the space of symmetric endomorphisms of these representations.

In the present section we describe how the space of symmetric endomorphisms of a real irreducible $SU(2)$ –module splits into irreducible components.

Let W be a \mathbf{C} –vector space with a Hermitian inner product. When the coefficient field on W is restricted to the field of real numbers, we denote by $W_{\mathbf{R}}$ the resulting \mathbf{R} –vector space with the complex structure J . The Hermitian inner product induces a symmetric inner product on $W_{\mathbf{R}}$, simply by taking the real part.

If $H(W)$ denotes the \mathbf{R} –vector space of Hermitian endomorphisms on W and $S(W_{\mathbf{R}})$ the \mathbf{R} –vector space of all symmetric endomorphisms on $W_{\mathbf{R}}$, it follows from general considerations above that $H(W) \subset S(W_{\mathbf{R}})$, while \mathbf{C} –linearity of $A \in H(W)$ is reflected in $S(W_{\mathbf{R}})$ by commutation of A and J .

Suppose that W has a real (resp. quaternionic) structure denoted by σ compatible with the Hermitian inner product. Then $H(W)$ has a regular action of σ such that $A \mapsto \sigma A \sigma$, where A is a Hermitian endomorphism. Hence, we can define the subspaces $H_{\pm}(W)$ of $H(W)$ as the set of invariant/anti-invariant Hermitian endomorphisms with respect to σ . The action of σ extends to $S(W_{\mathbf{R}})$ in the obvious way.

Lemma 5.1. *If $A \in H_+(W)$, then real endomorphisms σA and $J\sigma A$ are symmetric endomorphisms on $W_{\mathbf{R}}$.*

Proof. For simplicity, we assume that σ is a real strucutre. If σ is a quaternionic structure the proof goes along the same lines.

Let $A \in H_+(W)$ and so $\sigma A = A\sigma$.

Denote the Hermitian inner product on W by $(\ , \)$, with the convention in which it is \mathbf{C} –linear in the first argument, and let $\langle \ , \ \rangle$ be the induced symmetric

inner product on $W_{\mathbf{R}}$. Then, for $u, v \in W \cong W_{\mathbf{R}}$,

$$\begin{aligned}\langle \sigma Au, v \rangle &= \operatorname{Re}(\sigma Au, v) = \operatorname{Re}(\overline{Au, \sigma v}) = \operatorname{Re}(u, A\sigma v) \\ &= \operatorname{Re}(A\sigma v, u) = \operatorname{Re}(\sigma Av, u) = \langle \sigma Av, u \rangle\end{aligned}$$

Therefore, $\sigma A \in S(W_{\mathbf{R}})$. The proof for $J\sigma A$ is analogous. \square

Notice that σA (resp. $J\sigma A$) above is not an Hermitian operator since σ is by definition conjugate-linear. We put

$$\sigma H_+(W) := \{\sigma A \mid A \in H_+(W)\} \subset S(W_{\mathbf{R}}),$$

$$J\sigma H_+(W) := \{J\sigma A \mid A \in H_+(W)\} \subset S(W_{\mathbf{R}}).$$

A characterization of these subspaces is given as follows:

Lemma 5.2. *Let B be a symmetric endomorphism of $W_{\mathbf{R}}$. Then,*

- (1) *B belongs to $\sigma H_+(W)$ if and only if $JB = -BJ$ and $\sigma B\sigma = B$;*
- (2) *B belongs to $J\sigma H_+(W)$ if and only if $JB = -BJ$ and $\sigma B\sigma = -B$.*

Proof. For B in $\sigma H_+(W)$ (resp. in $J\sigma H_+(W)$) there exists $A \in H_+(W)$ such that $B = \sigma A$ (resp. $J\sigma A$). Write BJ , $\sigma B\sigma$ in terms of A ; commutation relations for A , J , σ yield the implications.

Conversely, condition $JB = -BJ$ implies that B is not Hermitian; hence, $A := \sigma B$ (resp. $A := J\sigma B$) is, for commutation relations between J, σ lead to $AJ = JA$. Invariance under the regular action of σ on $H(W)$ shows $A \in H_+(W)$, therefore B belongs to $\sigma H_+(W)$ (resp. $J\sigma H_+(W)$). \square

Subspaces $\sigma H_+(W)$ and $J\sigma H_+(W)$ are orthogonal with respect to the inherited inner product on $S(W_{\mathbf{R}})$. Then, counting dimensions we have:

Corollary 5.3. *We have a decomposition of $S(W_{\mathbf{R}})$:*

$$S(W_{\mathbf{R}}) = H_+(W) \oplus H_-(W) \oplus \sigma H_+(W) \oplus J\sigma H_+(W).$$

Remark 3. As a result, the orthogonal complement of $H(W)$ in $S(W_{\mathbf{R}})$ has the induced complex structure.

Applying Corollary 5.3 to the real representations of $SU(2)$ discussed in §4 yields:

Proposition 5.4.

$$\begin{aligned}H_+(S^{2k}\mathbf{C}^2) &= \bigoplus_{r=0}^k S_0^{2k-2r}\mathbf{R}^3, & H_-(S^{2k}\mathbf{C}^2) &= \bigoplus_{r=0}^{k-1} S_0^{2k-1-2r}\mathbf{R}^3 \\ H_+(S^{2k+1}\mathbf{C}^2) &= \bigoplus_{r=0}^k S_0^{2k+1-2r}\mathbf{R}^3, & H_-(S^{2k+1}\mathbf{C}^2) &= \bigoplus_{r=0}^k S_0^{2k-2r}\mathbf{R}^3.\end{aligned}$$

6. RIGIDITY OF THE REAL STANDARD MAP

Essential at this stage is to prove proposition 6.3 (and its real invariant counterpart proposition 6.5). This is a technical result that states in short that if each factor in the normal decomposition of a G -module W is inequivalent as a K -representation to any other factor, there is a certain G -orbit in $H(W)$ which contains all class-one representations of (G, K) . Since in our case $H(W)$ itself is composed of class-one representations only, the G -orbit mentioned earlier fills $H(W)$.

The proposition has a practical reading: the Hermitian/symmetric operators parametrising the moduli spaces belong to the orthogonal complement in $H(W)$ to the aforesaid G -orbit, but in the present situation this space is null. Therefore the induced map will be rigid. We use this information to study the real standard map, the outcome naming the section (theorem 6.4).

A detailed description of the normal decomposition can be found in [4]. Let us sketch the central ideas: Consider the situation described in §2, i.e., $W \subset \Gamma(V)$ is a space of sections of the vector bundle $V \rightarrow M$, $M = G/K$ associated to the principal homogeneous bundle $G \rightarrow G/K$ with standard fibre the irreducible K -representation $V_0 \subset W$. Furthermore, suppose $V \rightarrow M$ to be equipped with its canonical connection. Let $f : G/K \rightarrow Gr_p(W)$ be the corresponding induced map by $(V \rightarrow M, W)$. The space of sections W splits into V_0 and its orthogonal complement $N_0 = U_0$. Assume the condition of lemma 2.1, i.e., $\mathfrak{m}V_0 \subset U_0$ such that the canonical connection and the pull-back connection coincide.

From now on our considerations will be restricted at a point $o \in M$ for the sake of simplicity. The second fundamental form K at $o \in M$ is an element of $T_o^*M \otimes V_0^* \otimes U_0$ such that for all $X \in T_oM$, $v \in V_0$, $(K_X(v))_o \in U_0$. The image of this mapping, also designated by B_1 , is a well-defined subspace of N_0 and thus gives a further orthogonal decomposition of W as $V_0 \oplus \text{Im}B_1 \oplus (V_0 \oplus \text{Im}B_1)^\perp$. Call $N_1 = (V_0 \oplus \text{Im}B_1)^\perp$ the *first normal subspace*. Applying the connection to the second fundamental form at the point $o \in M$ we have $\nabla K \in S^2T_o^*M \otimes V_0^* \otimes U_0$ (where symmetrisation follows from Gauss–Codazzi equations and flatness of the connection on \underline{W}). If π_1 denotes the orthogonal projection $\pi_1 : W \rightarrow N_1$ then B_2 is defined as $\pi_1 \circ \nabla K \in S^2T_o^*M \otimes V_0^* \otimes N_1$, and we have $W = V_0 \oplus \text{Im}B_1 \oplus \text{Im}B_2 \oplus N_2$ where N_2 is the *second normal subspace*. Recursively, $B_p = \pi_{p-1} \circ \nabla^{p-1}K \in S^pT_o^*M \otimes V_0^* \otimes N_{p-1}$. This reiterative process leads to:

$$W = V_0 \oplus \text{Im} B_1 \oplus \text{Im} B_2 \oplus \cdots \oplus \text{Im} B_n \oplus N_n$$

If $N_n = 0$ this is called the *normal decomposition of W with respect to V_0* .

Let us enunciate without proof two results by the second-named author regarding the normal decomposition which are needed in the sequel to establish proposition 6.3.

Proposition 6.1. [10] *If W is an irreducible G -module, then for any K -module, $V_0 \subset W$ there exists a positive integer n such that $N_n = 0$, i.e.,*

$$(6.1) \quad W = V_0 \oplus \operatorname{Im} B_1 \oplus \cdots \oplus \operatorname{Im} B_n$$

which is a normal decomposition of (W, V_0) .

Proposition 6.2. [10] *Let W be a G -module and $V_0 \subset W$ a K -module. Suppose that (W, V_0) has a normal decomposition. Assume that each term in the decomposition (6.1) shares no common K -irreducible factor with any other term in the decomposition. Let T be a non-negative Hermitian endomorphism of W which satisfies $(Tgv_1, Tgv_2) = (v_1, v_2)$ for all $g \in G$, $v_1, v_2 \in V_0$. Then, if T is K -equivariant, $T = \operatorname{Id}_W$.*

A G -irreducible representation is a *class-one representation* of (G, K) , for K a closed subgroup of G (assumed compact), if it contains non-zero K -invariant elements. Then, we can state the following

Proposition 6.3. *Let $W = H^0(\mathbf{CP}^1, \mathcal{O}(k))$ and V_0 the K -representation regarded as the standard fibre for $\mathcal{O}(k) \rightarrow \mathbf{CP}^1$. Then, $\operatorname{GH}(V_0, V_0) = \operatorname{H}(W)$.*

Proof. By Borel–Weil theorem, W is identified with the $\operatorname{SU}(2)$ -representation $S^k \mathbf{C}^2$ and, using lemma 2.1, V_0 can be regarded as a subspace of W . The space W decomposes under the $\operatorname{U}(1)$ -action as

$$W|_{\operatorname{U}(1)} = \mathbf{C}_{-k} \oplus \mathbf{C}_{-k+2} \oplus \cdots \oplus \mathbf{C}_k.$$

Indeed, this is the normal decomposition by proposition 6.1 where $V_0 = \mathbf{C}_{-k}$.

Let H be a class-one submodule of (G, K) in $\operatorname{H}(W)$. Suppose that $H \not\subset \operatorname{GH}(V_0, V_0)$. Then, by a standard argument, we can assume that $H \perp \operatorname{GH}(V_0, V_0)$. Since H is a class-one representation, there exists a non-zero $C \in H$ such that $kCk^{-1} = C$ for all $k \in K$. It follows from the orthogonality assumption that

$$\begin{aligned} 0 &= (C, gH(v_1, v_2))_{\operatorname{H}(W)} = (C, H(gv_1, gv_2))_{\operatorname{H}(W)} \\ &= \frac{1}{2} \{ (Cgv_1, gv_2)_W + (Cgv_2, gv_1)_W \}, \end{aligned}$$

for arbitrary $g \in G$ and $v_1, v_2 \in V_0 \subset W$. Polarization gives

$$0 = (Cgv_1, gv_2), \quad g \in G, \quad v_1, v_2 \in V_0.$$

If C is sufficiently small, then $\operatorname{Id} + C > 0$ and so, we can define a positive Hermitian operator T satisfying $T^2 = \operatorname{Id} + C$. Then we have

$$(Tgv_1, Tgv_2) = (v_1, v_2) \quad g \in G, \quad v_1, v_2 \in V_0.$$

Since T is also K -equivariant, proposition 6.2 yields that $T = \operatorname{Id}$ and so, $C = 0$, which is a contradiction. Hence, every class-one subrepresentation of (G, K) in $\operatorname{H}(W)$ is included in $\operatorname{GH}(V_0, V_0)$. However, it follows from the Clesbsch–Gordan formulae that $\operatorname{H}(W)$ is composed by class-one representation of (G, K) only, therefore $\operatorname{GH}(V_0, V_0) = \operatorname{H}(W)$. \square

Remark 4. A more general version of proposition 6.3 can be found in [10], proposition 7.9. Our proof is essentially the same with the obvious particularizations.

We shall prove the following interesting result:

Theorem 6.4. *Let $W = S^{2k}\mathbf{C}^2$ such that $W^{\mathbf{R}} = S_0^k\mathbf{R}^3 \cong \mathbf{R}^{2k+1}$. If $f : \mathbf{CP}^1 \rightarrow \text{Gr}_{2k-1}(\mathbf{R}^{2k+1})$ is a holomorphic isometric embedding of degree $2k$, then f is the standard map by $W^{\mathbf{R}}$ up to gauge equivalence.*

Before proving theorem 6.4, let us clarify the construction of the mapping $f : \mathbf{CP}^1 \rightarrow \text{Gr}_{2k-1}(\mathbf{R}^{2k+1})$ from the vector bundle viewpoint.

If we regard the complex projective line as the symmetric space G/K where $G = \text{SU}(2)$ and $K = \text{U}(1)$, then by Borel–Weil theorem the space of sections $\Gamma(\mathcal{O}(2k))$ becomes a G -module such that $W = H^0(\mathbf{CP}^1; \mathcal{O}(2k)) \cong S^{2k}\mathbf{C}^2$. The decomposition of $S^{2k}\mathbf{C}^2$ into irreducible $\text{U}(1)$ -modules is as follows:

$$(6.2) \quad S^{2k}\mathbf{C}^2|_{\text{U}(1)} = \bigoplus_{r=0}^{2k} \mathbf{C}_{2k-2r}$$

The typical fibre of $\mathcal{O}(2k) \rightarrow \mathbf{CP}^1$ is regarded as a subspace \mathbf{C}_{-2k} in the decomposition by lemma 2.1.

Since W has an invariant real structure, we have an invariant real subspace denoted by $W^{\mathbf{R}} = (S^{2k}\mathbf{C}^2)^{\mathbf{R}} \cong S_0^k\mathbf{R}^3$ of real dimension $2k+1$. The real structure descends to the splitting (6.2) but now each irreducible $\text{U}(1)$ -module is not invariant under the real structure, but $\sigma(\mathbf{C}_{2k-2r}) = \mathbf{C}_{-2k+2r}$. Therefore for each $r = 0, \dots, k$ the space $(\mathbf{C}_{2k-2r} \oplus \mathbf{C}_{-2k+2r})$ is stable under the real structure and decomposes in two real isomorphic irreducible $\text{U}(1)$ -modules, denoted by $(\mathbf{C}_{2k-2r} \oplus \mathbf{C}_{-2k+2r})^{\mathbf{R}}$, such that (6.2) would be rewritten as

$$(6.3) \quad S_0^k\mathbf{R}^3|_{\text{U}(1)} = \bigoplus_{r=0}^k (\mathbf{C}_{2k-2r} \oplus \mathbf{C}_{-2k+2r})^{\mathbf{R}}.$$

This implies that $\mathcal{O}(2k) \rightarrow \mathbf{CP}^1$ is globally generated by $W^{\mathbf{R}}$. Thus, we can define a *real standard map* $f_0 : \mathbf{CP}^1 \rightarrow \text{Gr}_{2k-1}(\mathbf{R}^{2k+1})$ by $W^{\mathbf{R}}$, which turns out to be a holomorphic isometric embedding of degree $2k$ by lemma 2.3. Using the inner product on $W^{\mathbf{R}}$ and the fibre-metric on $\mathcal{O}(2k) \rightarrow \mathbf{CP}^1$, it is possible to define the adjoint of the evaluation which at the identity of G/K determines a mapping $ev_{[e]}^* : \mathcal{O}(2k) \rightarrow \underline{W}^{\mathbf{R}}$ whose image is just $(\mathbf{C}_{2k} \oplus \mathbf{C}_{-2k})^{\mathbf{R}}$.

Within this framework we have a real version of proposition 6.3, which is the in the core of the proof of theorem 6.4:

Proposition 6.5. *Let $W = H^0(\mathbf{CP}^1, \mathcal{O}(2k))$ and V_0 the K -representation regarded as the standard fibre for $\mathcal{O}(2k) \rightarrow \mathbf{CP}^1$. Then, $\text{GS}(V_0, V_0) = \text{S}(W^{\mathbf{R}})$.*

Proof. Equation (6.3) gives the normal decomposition of $W^{\mathbf{R}}$ where now $V_0 = (\mathbf{C}_{-2k} \oplus \mathbf{C}_{2k})^{\mathbf{R}}$. The space of symmetric endomorphisms of $W^{\mathbf{R}}$ can be identified

by decomposing first the tensor product using lemma 4.1, and identifying the symmetric components

$$S(W^{\mathbf{R}}) \subset \text{End}(W^{\mathbf{R}}) = W^{\mathbf{R}} \otimes_{\mathbf{R}} (W^{\mathbf{R}})^* \cong \otimes^2 W^{\mathbf{R}}$$

$$S(W^{\mathbf{R}}) = \bigoplus_{r=0}^k S_0^{4k-4r} \mathbf{R}^3 \subset \otimes^2 W^{\mathbf{R}} = \bigoplus_{r=0}^{2k} S_0^{4k-2r} \mathbf{R}^3$$

Notice that all these modules are class-one representations. Then, a similar argument as the one in the proof of proposition 6.3 yields the desired result. \square

We can now proceed to prove theorem 6.4:

Proof. Consider the real standard map by the holomorphic line bundle $\mathcal{O}(2k) \rightarrow \mathbf{CP}^1$ and $W^{\mathbf{R}}$ as depicted above. Therefore by proposition 6.5, $S(W^{\mathbf{R}}) = \text{GS}(V_0, V_0)$. and replacing \mathbf{R}^{n+2} by $W^{\mathbf{R}}$ in theorem 2.4 the real standard map admits no deformations. \square

7. MODULI SPACE BY GAUGE EQUIVALENCE

We undertake now the task of giving an accurate description of the moduli space of holomorphic isometric embeddings $\mathbf{CP}^1 \rightarrow Gr_p(W)$ up to gauge equivalence. Our strategy will be to capitalise on the representation-theoretic formulae of Sections 4 and 5 to explicitly determine the subspaces of linear operators in $S(W)$ which specify the moduli. Such subspaces are sharply characterised by condition (III) in theorem 2.4. This is achieved after a sequence of step-stone results culminating in lemma 7.2 and its Corollary, which allows to compute the moduli dimension.

As indicated by condition (IV) in theorem 2.4, the gauge equivalence relation is to be taken into account to obtain the moduli space and to give a geometric meaning to its compactification in the natural L^2 -topology. A qualitative description of these spaces is given in theorem 7.4.

Let W be the space of holomorphic sections of $\mathcal{O}(k) \rightarrow \mathbf{CP}^1$ which, by Borel–Weil theorem, is identified with the $\text{SU}(2)$ -representation $S^k \mathbf{C}^2$. Equation (6.2) gives a weight decomposition of W with respect to $\text{U}(1)$. When $\mathcal{O}(k) \rightarrow \mathbf{CP}^1$ is regarded as the homogeneous line bundle $\text{SU}(2) \times_{\text{U}(1)} V_0 \rightarrow \mathbf{CP}^1$, then V_0 is identified with the $\text{U}(1)$ -irreducible subspace \mathbf{C}_{-k} of W by lemma 2.1.

In order to apply theorem 2.4 we shall regard the quotient bundle as a real vector bundle of rank 2. Following the generalizaion of do Carmo–Wallach theory, we must determine the subspaces $\text{GS}(V_0, V_0)$ and $\text{GS}(\mathfrak{m}V_0, V_0)$ of $S(W)$.

From now on V_0 and W shall stand either for the complex modules or for their underlying \mathbf{R} -vector spaces whenever the meaning is clear, avoiding the heavier notation $(V_0)_{\mathbf{R}}$ or $W_{\mathbf{R}}$; In the remaining sections we will adopt this convention.

Since $\text{GH}(V_0, V_0)$ is a proper subspace of $\text{GS}(V_0, V_0)$, we have that $\text{H}(W) \subset \text{GS}(V_0, V_0)$. We must determine the intersection between $\text{GS}(V_0, V_0)$ and subspaces $\sigma\text{H}_+(W) \oplus J\sigma\text{H}_+(W)$ appearing in Corollary 5.3. The same is true for the intersection $\text{GS}(\mathfrak{m}V_0, V_0)$ with $\sigma\text{H}_+(W) \oplus J\sigma\text{H}_+(W)$ as we shall consider immediately.

Lemma 7.1. $\mathfrak{m}V_0 = \mathbf{C}_{-k-2}$.

Proof. By the decomposition of $S^2\mathbf{C}^2$ into $\text{U}(1)$ -irreducible modules $S^2\mathbf{C}^2|_{\text{U}(1)} = \mathbf{C}_2 \oplus \mathbf{C}_0 \oplus \mathbf{C}_{-2}$ and using the real structure we have $(S^2\mathbf{C}^2)^{\mathbf{R}} \cong \mathfrak{su}(2)$, $(\mathbf{C}_0)^{\mathbf{R}} \cong \mathfrak{u}(1)$ therefore $(\mathbf{C}_2 \oplus \mathbf{C}_{-2})^{\mathbf{R}} \cong \mathfrak{m}$. Then,

$$\mathfrak{m} \otimes V_0 = (\mathbf{C}_2 \oplus \mathbf{C}_{-2}) \otimes \mathbf{C}_{-k} = \mathbf{C}_{-k+2} \oplus \mathbf{C}_{-k-2}$$

The action of \mathfrak{m} on V_0 is then obtained by projecting $\mathfrak{m} \otimes V_0$ back to $S^k\mathbf{C}^2$; therefore

$$\mathfrak{m}V_0 = (\mathfrak{m} \otimes V_0) \cap S^k\mathbf{C}^2|_{\text{U}(1)} = \mathbf{C}_{-k+2}$$

□

Lemma 7.2. $\text{GS}(\mathfrak{m}V_0, V_0) \cap \sigma\text{H}_+(W) \oplus J\sigma\text{H}_+(W)$ is the highest weight representations of $SU(2)$ appeared in proposition 5.4.

Proof. Let u_{-k} and u_{-k+2} be respectively unitary bases for the complex one-dimensional $\text{U}(1)$ -modules $V_0 = \mathbf{C}_{-k}$ and $\mathfrak{m}V_0 = \mathbf{C}_{-k+2}$. Then, The space $\text{H}(\mathfrak{m}V_0, V_0) \equiv \text{H}(\mathbf{C}_{-k+2}, \mathbf{C}_{-k})$ is the complex span of

$$2\text{H}(u_{-k+2}, u_{-k}) = u_{-k+2} \otimes (\cdot, u_{-k})_W + u_{-k} \otimes (\cdot, u_{-k+2})_W$$

where $(\cdot, \cdot)_W$ denotes the Hermitian inner product on $S^k\mathbf{C}^2$. When \mathbf{C}_{-k} and \mathbf{C}_{-k+2} are regarded as their underlying two-dimensional \mathbf{R} -vector spaces \mathbf{R}_k^2 and \mathbf{R}_{k-2}^2 , real bases are given respectively by $\{u_{-k}, Ju_{-k}\}$ and $\{u_{-k+2}, Ju_{-k+2}\}$ where J is the almost complex structure induced by the multiplication by the imaginary unit. Using these real bases the complex form $2\text{H}(u_{-k+2}, u_{-k})$ can be rewritten as a real operator

$$\begin{aligned} 2\text{H}(u_{-k+2}, u_{-k})|_{\mathbf{R}} &= u_{-k+2} \otimes \langle \cdot, u_{-k} \rangle_W + Ju_{-k+2} \otimes \langle \cdot, Ju_{-k} \rangle_W \\ &\quad + u_{-k} \otimes \langle \cdot, u_{-k+2} \rangle_W + Ju_{-k} \otimes \langle \cdot, Ju_{-k+2} \rangle_W \end{aligned}$$

where $\langle \cdot, \cdot \rangle_W$ is the inner product on $W_{\mathbf{R}}$ induced from the Hermitian inner product on W . Write the basis for $\text{S}(\mathfrak{m}V_0, V_0) \equiv \text{S}(\mathbf{R}_{-k+2}^2, \mathbf{R}_{-k}^2)$ as $\{\text{S}(u_{-k+2}, u_{-k}), \text{S}(Ju_{-k+2}, u_{-k}), \text{S}(u_{-k+2}, Ju_{-k}), \text{S}(Ju_{-k+2}, Ju_{-k})\}$, eg.,

$$2\text{S}(u_{-k+2}, u_{-k}) = u_{-k+2} \otimes \langle \cdot, u_{-k} \rangle_W + u_{-k} \otimes \langle \cdot, u_{-k+2} \rangle_W, \quad \text{etc.}$$

Comparing both equations we have:

$$\text{H}(u_{-k+2}, u_{-k})|_{\mathbf{R}} = \text{S}(u_{-k+2}, u_{-k}) + \text{S}(Ju_{-k+2}, Ju_{-k}).$$

Analogously,

$$\text{H}(u_{-k+2}, iu_{-k})|_{\mathbf{R}} = \text{S}(u_{-k+2}, Ju_{-k}) - \text{S}(Ju_{-k+2}, u_{-k}).$$

Let us define a new elements $\{X, Y\}$

$$\begin{aligned} X &= S(u_{-k+2}, u_{-k}) - S(Ju_{-k+2}, Ju_{-k}) \\ Y &= S(u_{-k+2}, Ju_{-k}) + S(Ju_{-k+2}, u_{-k}) \end{aligned}$$

$X, Y \in S(W_{\mathbf{R}})$ are orthogonal to the subspace of Hermitian matrices $H(W) \subset S(W_{\mathbf{R}})$, therefore they belong to $\sigma H_+(W) \oplus J\sigma H_+(W)$ according to Corollary 5.3.

Let us consider the contragredient action of the structure map σ on X , the case of Y being analogous. Firstly,

$$\sigma(u \otimes \langle \cdot, v \rangle_W) \sigma = \sigma u \otimes \langle \sigma \cdot, v \rangle_W = \sigma u \otimes \langle \cdot, \sigma v \rangle_W$$

and as such $\sigma S(u, v) \sigma = S(\sigma u, \sigma v)$.

Secondly, the $U(1)$ -modules \mathbf{C}_i are not σ -invariant but $\sigma(\mathbf{C}_{\pm i}) = \mathbf{C}_{\mp i}$, for all i that is, $\sigma u_{\pm i} = u_{\mp i}$. which, together with conjugate-linearity of the structure map yields $\sigma(\mathbf{R}_{\pm i}^2) = \mathbf{R}_{\mp i}^2 : \{u_{\pm i}, Ju_{\pm i}\} \mapsto \{u_{\mp i}, -Ju_{\mp i}\}$. Hence we have:

$$\begin{aligned} X^\sigma = \sigma X \sigma &= S(\sigma u_{-k+2}, \sigma u_{-k}) - S(\sigma Ju_{-k+2}, \sigma Ju_{-k}) \\ &= S(u_{k-2}, u_k) - S(Ju_{k-2}, Ju_k) \end{aligned}$$

This is not an element of $S(\mathfrak{m}V_0, V_0) \equiv S(\mathbf{R}_{-k+2}^2, \mathbf{R}_{-k}^2)$ but $X^\sigma \in S(\mathbf{R}_{k-2}^2, \mathbf{R}_k^2)$. Note that we can find $g \in SU(2)$ such that $S(u_{k-2}, u_k) = S(gu_{-k+2}, gu_{-k}) = g \cdot S(u_{-k+2}, u_{-k}) \in GS(\mathfrak{m}V_0, V_0)$ up to a sign.

Let us add $Y^\sigma = S(u_{k-2}, Ju_k) + S(Ju_{k-2}, u_k)$ for the sake of completeness. The preceding argument also shows that $\{S(u_{k-2}, u_k), S(u_{k-2}, Ju_k), S(Ju_{k-2}, u_k), S(Ju_{k-2}, Ju_k)\}$ spans a subspace of $GS(\mathfrak{m}V_0, V_0)$.

Moreover, using the characterisation given in Corollary 5.3 we have:

$$X + X^\sigma \in \sigma H_+(W), \quad X - X^\sigma \in J\sigma H_+(W)$$

The same inclusions are also true for $Y \pm Y^\sigma$.

From the expression of the action of σ on $H(u, v)$

$$\begin{aligned} \sigma \cdot H(u, v) &= \sigma(u \otimes \langle \cdot, v \rangle_W + v \otimes \langle \cdot, u \rangle_W) \sigma = \sigma u \otimes \langle \sigma \cdot, v \rangle_W + \sigma v \otimes \langle \sigma \cdot, u \rangle_W \\ &= \sigma u \otimes \overline{\langle \cdot, \sigma v \rangle_W} + \sigma v \otimes \overline{\langle \cdot, \sigma u \rangle_W} \end{aligned}$$

it is easy to write $X \pm X^\sigma$ back in terms of Hermitian operators as

$$X \pm X^\sigma = \sigma \cdot (H(u_{k-2}, u_k) \pm H(u_{-k+2}, u_{-k})) | \mathbf{R}$$

The toral action of a $U(1)$ -element of $SU(2)$ on $u_{\pm k}, u_{\pm(k-2)}$ yields

$$\exp(i\theta)u_{\pm k} = \exp(\pm ik\theta)u_{\pm k}, \quad \exp(i\theta)u_{\pm(k-2)} = \exp(\pm i(k-2)\theta)u_{\pm(k-2)}$$

and as such, $X \pm X^\sigma$ (considered as the Hermitian operator above) contains terms of weight $\pm(2k-2)$. However, from Corollary 5.4 we know that the only component in the real decomposition of $\sigma H_+(W)$ and $J\sigma H_+(W)$ (both isomorphic

to $H_+(W)$) which can host such a vector is the top term $S_0^k \mathbf{R}^3$ on each space. Therefore

$$\mathrm{GS}(\mathfrak{m}V_0, V_0) \cap \sigma H_+(W) = S_0^k \mathbf{R}^3 \quad (\text{resp. for } J\sigma H_+(W))$$

And as a result

$$\mathrm{GS}(\mathfrak{m}V_0, V_0) = H(W) \oplus S_0^k \mathbf{R}^3 \oplus S_0^k \mathbf{R}^3$$

□

In other words:

Corollary 7.3. *The orthogonal complement to $\mathrm{GS}(\mathfrak{m}V_0, V_0) \oplus \mathbf{R} \mathrm{Id}$ in $S(W)$ is*

$$2 \bigoplus_{r=1}^{k \geq 2r} S_0^{k-2r} \mathbf{R}^3$$

This follows from applying the previous lemma to the explicit expressions for the components of $S(W)$ as described in proposition 5.4, and accounts for the space of symmetric operators T described by the second relation in (2.1), i.e., condition III in theorem 2.4.

Remark 5. The first condition in (2.1) is for all our purposes inessential: Let $\mathrm{GS}_0(V_0, V_0)$ be the orthogonal complement of the G -invariant, irreducible submodule generated by the identity in $\mathrm{GS}(V_0, V_0)$. We denote by $S_0(W)$ the set of tracefree symmetric operators on W with the induced inner product from $S(W)$. Then,

$$\mathrm{GS}_0(V_0, V_0) \subset \mathrm{GS}(\mathfrak{m}V_0, V_0),$$

which stems from an analogous result to lemma 7.2 applied to $\mathrm{GS}_0(V_0, V_0)$. The proof is equivalent, changing the weight $\pm(2k-2)$ by $\pm 2k$ in the curcial final step.

Condition III in theorem 2.4 is fulfilled by the family of operators in Corollary 7.3 (see remark above) thus accounting for all holomorphic embeddings $f : \mathbf{CP}^1 \rightarrow Gr_p(\mathbf{R}^m)$ up to possible degeneracies. Quantitative information about the moduli (i.e., its dimension) can therefore be derived from the Corollary:

$$(7.1) \quad \dim_{\mathbf{R}} \mathcal{M}_k = k(k-1).$$

The following theorem summarises the qualitative information about the moduli space and gives a neat geometric interpretation to its compactification.

Theorem 7.4. *If $f : \mathbf{CP}^1 \rightarrow Gr_n(\mathbf{R}^{n+2})$ is a full holomorphic isometric embedding of degree k , then $n \leq 2k$.*

Let \mathcal{M}_k be the moduli space of full holomorphic isometric embeddings of degree k of the complex projective line into $Gr_{2k}(\mathbf{R}^{2k+2})$ by the gauge equivalence of maps. Then, \mathcal{M}_k can be regarded as an open bounded convex body in $2 \bigoplus_{r=1}^{k \geq 2r} S_0^{k-2r} \mathbf{R}^3$.

Let $\overline{\mathcal{M}_k}$ be the closure of the moduli \mathcal{M}_k by the inner product. boundary points of $\overline{\mathcal{M}_k}$ describe those maps whose images are included in some totally geodesic submanifold $Gr_p(\mathbf{R}^{p+2})$ of $Gr_{2k}(\mathbf{R}^{2k+2})$, where $p < 2k$.

The totally geodesic submanifold $Gr_p(\mathbf{R}^{p+2})$ can be regarded as the common zero set of some sections of $Q \rightarrow Gr_{2k}(\mathbf{R}^{2k+2})$, which belongs to \mathbf{R}^{2k+2} .

Proof. The restriction $n \leq 2k$ follows from I in theorem 2.4 and Borel–Weil theorem.

It is evident from III in theorem 2.4 that $GS(\mathfrak{m}V_0, V_0)^\perp$ is a parametrisation of the space of full holomorphic isometric embeddings $f : \mathbf{CP}^1 \rightarrow Gr_{2k}(\mathbf{R}^{2k+2})$ of degree k . Positivity of T being guaranteed by fullness, we can apply the original do Carmo–Wallach argument [4], §5.1, to conclude that \mathcal{M}_k is a bounded connected open convex body in $H(W)$ with the topology induced by the L^2 scalar product.

Under the natural compactification in the L^2 -topology, the boundary points correspond to operators T which are not positive, but semipositive. It follows from IV in theorem 2.4 that each of these operators defines in turn a full holomorphic isometric embedding $\mathbf{CP}^1 \rightarrow Gr_p(\mathbf{R}^{p+2})$, of degree k with $p = 2k - \dim \text{Ker } T$, whose target embeds in $Gr_{2k}(\mathbf{R}^{2k+2})$ as a totally geodesic submanifold. The image Z of the embedding $Gr_p(\mathbf{R}^{p+2}) \rightarrow Gr_{2k}(\mathbf{R}^{2k+2})$ is determined by the common zero-set of sections in $\text{Ker } T$. \square

8. MODULI SPACE BY IMAGE EQUIVALENCE

Remember that $S_0^k \mathbf{R}^3$, defined in lemma 4.1, is the real invariant part of $S^{2k} \mathbf{C}^2$ which is the symmetric power of the standard representation \mathbf{C}^2 of $SU(2)$. The moduli space \mathcal{M}_k has a natural complex structure induced by that on $Q \rightarrow Gr_{2k}(\mathbf{R}^{2k+2})$ which coincides with the one introduced in Remark 3, §5. Hence, \mathcal{M}_k can be regarded as holomorphically included in the \mathbf{C} -vector space $\bigoplus_{r=1}^{k \geq 2r} S^{2k-4r} \mathbf{C}^2$. We can show that the centralizer of the holonomy group acts on \mathcal{M}_k with weight $-k$. Hence we have

Theorem 8.1. *Let \mathbf{M}_k be the moduli space of holomorphic isometric embeddings of the complex projective line into $Gr_{2k}(\mathbf{R}^{2k+2})$ of degree k by the image equivalence of maps. Then we have $\mathbf{M}_k = \mathcal{M}_k / S^1$.*

Proof. Assume two full holomorphic isometric embeddings $\mathbf{CP}^1 \rightarrow Gr_{2k}(\mathbf{R}^{2k+2})$ of degree k to be image equivalent. They may represent distinct points in \mathcal{M}_k . By definition of image equivalence, there is an isometry ψ of $Gr_{2k}(\mathbf{R}^{2k+2})$ such that $f_2 = \psi \circ f_1$, then $f_2^* Q = f_1^* \tilde{\psi} Q$ as sets. Using the natural identifications ϕ_1, ϕ_2 of IV in theorem 2.4 we introduce new bundle isomorphisms $\mathcal{O}(k) \rightarrow f_2^* Q$ defined by $\tilde{\psi} \circ \phi_1$ and ϕ_2 . Hence, we have a gauge transformation $\phi_2^{-1} \tilde{\psi} \phi_1$ on the line bundle $\mathcal{O}(k) \rightarrow M$ preserving the metric and the connection. By connectedness of \mathbf{CP}^1 such a gauge transformation is regarded as an element of the centralizer of the holonomy group of the connection in the structure group of V i.e., $U(1) \equiv S^1$

acting with weight $-k$ on the standard fibre $V_0 \cong \mathbf{C}_{-k}$. Modding out the S^1 -action yields the true moduli space by image equivalence \mathbf{M}_k . \square

Remark 6. \mathcal{M}_k has a complex structure (see remark in §5) and a metric induced by the inner product both preserved by the S^1 -action. Hence, it is a Kähler manifold together with a S^1 -action preserving the Kähler structure. Therefore, \mathcal{M}_k is naturally equipped with a moment map $\mu : \mathcal{M}_k \rightarrow \mathbf{R}$ expressed as $\mu = |T|^2$.

Corollary 8.2. *There exists a one-parameter family $\{f_t\}$, $t \in [0, 1]$, of $\mathrm{SU}(2)$ -equivariant image-inequivalent holomorphic isometric embeddings of even degree of \mathbf{CP}^1 into complex quadrics where f_0 corresponds to the standard map and f_1 is the real standard map.*

Proof. The moduli space by gauge-equivalence \mathcal{M}_k is contained in $\oplus_{r=1}^{k \geq 2r} S^{2k-4r} \mathbf{C}^2$. For even k this last expression includes the trivial representation \mathbf{C} , which using the real structure can be described as $\mathbf{C} = \mathbf{R}\sigma \oplus \mathbf{R}J\sigma$. Let $C \in \mathbf{C} \subset \oplus_{r=1}^{k \geq 2r} S^{2k-4r} \mathbf{C}^2$. If it is small enough, then by theorem 2.4, $Id + C$ determines a holomorphic isometric embedding into $Gr_{2k}(\mathbf{R}^{2k+2})$. The group $\mathrm{SU}(2)$ acts on each component of $Id + C$ trivially, so the associated holomorphic isometric embedding is $\mathrm{SU}(2)$ -equivariant. The S^1 action of the centralizer of the holonomy group acts on \mathbf{C} with weight $-k$ (see proof of theorem 8.1) therefore, taking quotient by the S^1 -action, we obtain a half-open segment parametrising the described maps, which becomes a closed segment under the natural compactification in the L^2 -topology. Let $C = t\sigma + sJ\sigma$. Then we can show that $Id + C$ is positive if and only if $t^2 + s^2 < 1$. Suppose that $t^2 + s^2 = 1$. Then $(t + sJ)\sigma$ is also an invariant real structure on $S^{2k} \mathbf{C}^2$. Hence we may consider only the case that $t = 1$ and $s = 0$. Since the kernel of $Id + \sigma$ is $JW^{\mathbf{R}}$, theorem 2.4 implies that $Id + \sigma$ determines a totally geodesic submanifold $Gr_{2k-1}(\mathbf{R}^{2k+1})$ of $Gr_{4k}(\mathbf{R}^{4k+2})$ and a holomorphic isometric embedding into the submanifold $Gr_{2k-1}(\mathbf{R}^{2k+1})$ represented by $2Id_{W^{\mathbf{R}}}$. This map is nothing but the real standard map by $W^{\mathbf{R}}$, because constant multiple of the identity gives the same subspace of $W^{\mathbf{R}}$. \square

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(OM) DEPARTMENT OF GEOMETRY AND TOPOLOGY, FACULTY OF MATHEMATICAL SCIENCES, UNIVERSITY OF VALENCIA, C.DR MOLINER, 50, BURJASSOT, 46100, VALENCIA, SPAIN

E-mail address: oscar.macia@uv.es

(YN) DEPARTMENT OF MATHEMATICS, MEIJI UNIVERSITY, HIGASHI-MITA, TAMA-KU, KAWASAKI-SHI, KANAGAWA 214-8571, JAPAN

E-mail address: yasunaga@meiji.ac.jp

(MT) DEPARTMENT OF GENERAL EDUCATION, KURUME NATIONAL COLLEGE OF TECHNOLOGY, KURUME, FUKUOKA 830-8555, JAPAN.

E-mail address: masaro@GES.kurume-nct.ac.jp